SF 298 MASTER COPY KEEP THIS COPY FOR REPRODUCTION PURPOSES

REPORT DOCUMENTATION PAGE

Form Approved
OMB NO. 0704-0188

1. AGENCY USE ONLY *(Leave blank)*

2. REPORT DATE
October 2000

3. REPORT TYPE AND DATES COVERED
Technical Reports; October 2000

4. TITLE AND SUBTITLE
Optimal rate of empirical Bayes tests for lower truncation parameters

5. FUNDING NUMBERS
DAAD 19-00-1-0502

6. AUTHOR(S)
Shanti S. Gupta and Jianjun Li

7. PERFORMING ORGANIZATION NAMES(S) AND ADDRESS(ES)
Purdue University
Department of Statistics
West Lafayette, IN 47907-1399

8. PERFORMING ORGANIZATION REPORT NUMBER
Technical Report #00-07

9. SPONSORING / MONITORING AGENCY NAME(S) AND ADDRESS(ES)
U.S. Army Research Office
P.O. Box 12211
Research Triangle Park, NC 27709-2211

10. SPONSORING / MONITORING AGENCY REPORT NUMBER
ARO 40940.1-MA

11. SUPPLEMENTARY NOTES
The views, opinions and/or findings contained in this report are those of the author(s) and should not be construed as an official Department of the Army position, policy or decision, unless so designated by other documentation.

12a. DISTRIBUTION / AVAILABILITY STATEMENT

Approved for public release; distribution unlimited.

12 b. DISTRIBUTION CODE

13. ABSTRACT *(Maximum 200 words)*
The distributions with lower truncation parameters are important models in statistics and have been studied in recent years. In this paper, we consider the one-sided testing problem for lower truncation parameters through the empirical Bayes approach. The optimal rate of the monotone empirical Bayes tests is obtained and a monotone empirical Bayes test δ_n achieving the optimal rate is constructed. It is shown that δ_n has good performance for both small samples and large samples.

20010301 137

14. SUBJECT TERMS
Empirical Bayes, regret Bayes risk, optimal rate of convergence, minimax.

15. NUMBER IF PAGES
13

16. PRICE CODE

17. SECURITY CLASSIFICATION OR REPORT	18. SECURITY CLASSIFICATION OF THIS PAGE	19. SECURITY CLASSIFICATION OF ABSTRACT	20. LIMITATION OF ABSTRACT
UNCLASSIFIED	UNCLASSIFIED	UNCLASSIFIED	UL

NSN 7540-01-280-5500

Standard Form 298 (Rev. 2-89)
Prescribed by ANSI Std. 239-18

OPTIMAL RATE OF EMPIRICAL BAYES TESTS FOR LOWER TRUNCATION PARAMETERS

by

Shanti S. Gupta and Jianjun Li
Purdue University Purdue University

Technical Report # 00-07

Department of Statistics
Purdue University
West Lafayette, IN USA

October 2000

OPTIMAL RATE OF EMPIRICAL BAYES TESTS FOR LOWER TRUNCATION PARAMETERS [1]

Shanti S. Gupta
Department of Statistics
Purdue University
W. Lafayette, IN 47907

Jianjun Li
Department of Statistics
Purdue University
W. Lafayette, IN 47907

Abstract: The distributions with lower truncation parameters are important models in statistics and have been studied in recent years. In this paper, we consider the one-sided testing problem for lower truncation parameters through the empirical Bayes approach. The optimal rate of the monotone empirical Bayes tests is obtained and a monotone empirical Bayes test δ_n achieving the optimal rate is constructed. It is shown that δ_n has good performance for both small samples and large samples.

MS Classification: 62C12.

Keywords: Empirical Bayes, regret Bayes risk, optimal rate of convergence, minimax.

[1]This research was supported in part by US Army Research Office, Grant DAAD19-00-1-0502 at Purdue University.

1. Introduction. Let X denote a random variable having density function

$$f(x|\theta) = a(x)/A(\theta), \qquad \theta \leq x < b \leq \infty, \tag{1.1}$$

where $a(x)$ is a positive, continuous function on $(0, b)$, $A(\theta) = \int_\theta^b a(x)dx < \infty$ for every $\theta > 0$, θ is the parameter, which is distributed according to an unknown prior distribution G on $(0, b)$. Two typical examples of (1.1) are: (I) the exponential distribution with a location parameter: $f(x|\theta) = e^{-(x-\theta)}$, $x \geq \theta$, and (II) the Pareto distribution: $f(x|\theta) = \alpha\theta^\alpha / x^{\alpha+1}$, $x \geq \theta$.

We consider the problem of testing the hypotheses $H_0 : \theta \leq \theta_0$ versus $H_1 : \theta > \theta_0$, where θ_0 is known and $0 < \theta_0 < b$. The loss function is $l(\theta, 0) = \max\{\theta - \theta_0, 0\}$ for accepting H_0 and $l(\theta, 1) = \max\{\theta_0 - \theta, 0\}$ for accepting H_1. A test $\delta(x)$ is defined to be a measurable mapping from $(0, \infty)$ into $[0, 1]$ so that $\delta(x) = P\{$ accepting $H_1 | X = x\}$, i.e., $\delta(x)$ is the probability of accepting H_1 when $X = x$ is observed. Let $R(G, \delta)$ denote the Bayes risk of the test δ when G is the prior distribution. Given that $\int_0^\infty \theta dG(\theta) < \infty$, a Bayes test δ_G is found as

$$\delta_G(x) = 1 \quad \text{if} \quad E[\theta|X = x] \geq \theta_0; \quad \delta_G(x) = 0 \quad \text{if} \quad E[\theta|X = x] < \theta_0. \tag{1.2}$$

Because $E[\theta|X = x]$ involves G, the above solution works only if the prior G is given. If G is unknown, this testing problem is formed as a compound decision problem and the empirical Bayes approach is used. Let $X_1, X_2, \cdots, X_n$ be the observations from n independent past experiences. Based on $\widetilde{X}_n = (X_1, X_2, \cdots, X_n)$ and X, an empirical Bayes rule $\delta_n(X, \widetilde{X}_n)$ can be constructed. The performance of δ_n is measured by $R(G, \delta_n) - R(G, \delta_G)$, where $R(G, \delta_n) = E[R(G, \delta_n | \widetilde{X}_n)]$. The quantity $R(G, \delta_n) - R(G, \delta_G)$ is referred as the regret Bayes risk (or regret) in the literature.

This empirical Bayes approach was introduced by Robbins (1956, 1964). Since then, it has been widely used in statistics. For the family (1.1), some problems of statistical

inference based on the empirical Bayes method have been considered by Prasad and Singh (1990), Liang (1993), Datta (1994), Huang (1995), Balakrishnan and Ma (1997), Huang and Liang (1997), Ma and Balakrishnan (2000), among others. In this paper, we consider the testing problem and study the empirical Bayes tests for the family (1.1). The optimal rate of convergence of monotone empirical Bayes tests is obtained and a test with the optimal rate is constructed.

The paper is organized as follows. In Section 2 we provide a few preliminary results. In Section 3 we construct a monotone empirical Bayes test δ_n and obtain an upper bound of its regret. In Section 4, a minimax lower bound of the regrets of monotone empirical Bayes tests is obtained. Since the rates in the upper bound of Section 3 and lower bound of Section 4 coincide, the optimal rate is identified. As a byproduct, we see that δ_n achieves the optimal rate of convergence. The proofs of main results in Section 3 and 4 are given in Section 5.

2. Preliminary. We assume that $P(\theta > \theta_0) \cdot P(\theta < \theta_0) > 0$ in this paper. If $P(\theta > \theta_0) = 0$ or $P(\theta < \theta_0) = 0$, we know which action we should take regardless of the value of x. For example, if $P(\theta < \theta_0) = 0$, we accept H_1 always. So both two cases are excluded in the testing problem. We also assume $\mu_G \equiv \int_0^\infty \theta dG(\theta) < \infty$ so that the Bayes analysis can be carried out.

Let $f_G(x) = \int f(x|\theta)dG(\theta)$ be the marginal pdf of X, and $\phi_G(x) = E[\theta|X = x]$ be the posterior mean of θ given $X = x$. Note that $\phi_G(x)$ is increasing and $\phi_G(\theta_0) < \theta_0$. Then the Bayes rule stated in Section 1 can be represented as

$$\delta_G(x) = 1 \text{ if } \phi_G(x) \geq \theta_0 \Longleftrightarrow x \geq c_G; \quad \delta_G(x) = 0 \text{ if } \phi_G(x) < \theta_0 \Longleftrightarrow x < c_G.$$

where $c_G = \inf\{x \in (\theta_0, b) : \phi_G(x) \geq \theta_0\}$. c_G is called the critical point corresponding to G.

Since the Bayes rule δ_G is characterized by a single number c_G, a monotone empirical

Bayes test (MEBT) can be constructed through estimating c_G by $c_n(X_1, X_2, \cdots, X_n)$, say, and defining

$$\delta_n = \begin{cases} 1 & \text{if} \quad x \geq c_n, \\ 0 & \text{if} \quad x < c_n. \end{cases} \tag{2.1}$$

Then the regret of δ_n is

$$R(G, \delta_n) - R(G, \delta_G) = E \int_{c_n}^{c_G} w(x) a(x) dx, \tag{2.2}$$

where $w(x) = \alpha_G(x)[\theta_0 - \phi_G(x)]$ and $\alpha_G(x) = \int_{(0,x]} dG(\theta)/A(\theta)$.

To consider the rate of convergence of $R(G, \delta_n) - R(G, \delta_G)$, we assume that for some $r \geq 1$, $\alpha_G(x)$ is r-times continuously differentiable and for $i = 0, 1, \cdots, r$,

$$\sup_{\theta_0/2 < x < b} |\alpha_G^{(i)}(x)| \leq B_r < \infty. \tag{2.3}$$

Furthermore, we assume that

$$g(c_G) = G'(c_G) \neq 0. \tag{2.4}$$

From (2.3), we know that $\phi_G(x)$ is continuous. Then $c_G > \theta_0$ since $\phi_G(\theta_0) < \theta_0$. Also, from (2.3) and (2.4), $\theta_0 < b$ since $A(b-) = \lim_{x \uparrow b} A(x) = 0$.

3. An upper bound. We shall construct a MEBT and find an upper bound of its regret. The kernel method is used in the construction. Let $K_0(y)$ be a Borel-measurable, bounded function vanishing outside the interval $[0, 1]$ such that

$$\int_0^1 y^j K_0(y) dy = \begin{cases} 1 & \text{if} \quad j = 0, \\ 0 & \text{if} \quad j = 1, 2, \cdots, r. \end{cases} \tag{3.1}$$

Suppose $|K_0(y)| \leq B$. Denote $K_1(y) = \int_0^y K_0(s) ds$ and $u_n = n^{-1/(2r+1)}$. For any $x \in (0, b)$, define

$$W_n(x) = \frac{\theta_0 - x}{n u_n} \sum_{j=1}^{n} \frac{K_0(\frac{x - X_j}{u_n})}{a(X_j)} - \frac{1}{n} \sum_{j=1}^{n} \frac{K_1(\frac{x - X_j}{u_n})}{a(X_j)}. \tag{3.2}$$

It is shown later that $W_n(x)$ is a consistent estimator of $w(x)$. Since $P(\theta < \theta_0) > 0$, $\alpha_G(x) > 0$ for $x > \theta_0$. Thus $c_G = \int_{\theta_0}^{b} I_{[w(x)>0]}dx + \theta_0$. Let

$$c_n = \int_{\theta_0}^{d_n} I_{[W_n(x)>0]}dx + \theta_0, \tag{3.3}$$

where

$$d_n = \begin{cases} (\theta_0 + \ln n) \wedge b & \text{if} \quad a(b-) > 0, \\ \inf\{x \geq \theta_0 : a(x) < {u_n}^{1/3}\} \wedge (\theta_0 + \ln n) \wedge b & \text{if} \quad a(b-) = 0. \end{cases}$$

Then we propose a monotone empirical Bayes test $\delta_n(x)$ by

$$\delta_n = \begin{cases} 1 & \text{if} \quad x \geq c_n, \\ 0 & \text{if} \quad x < c_n. \end{cases} \tag{3.4}$$

Note that $d_n \to b$. As $d_n > c_G$,

$$c_n - c_G = -\int_{\theta_0}^{c_G} I_{[W_n(x)\leq 0]}dx + \int_{c_G}^{d_n} I_{[W_n(x)>0]}dx. \tag{3.5}$$

Note that δ_n is a monotone rule. It has good performance for small samples (See Van Houwelingen (1976)). Next we show that δ_n is a good procedure not only for small samples but also for large samples.

From (2.3), $w'(x)$ is continuous and $w'(x) = g(x)(\theta_0 - x)/A(x)$. Since $g(c_G) \neq 0$ and $c_G > \theta_0$, $w'(c_G) < 0$. Then $w'(x) < 0$ in a neighbourhood of c_G. For $\epsilon > 0$, define $A_\epsilon \equiv \min\{-w'(x) : x \in [c_G - \epsilon, c_G + \epsilon]\}$. Suppose $\epsilon_G > 0$ such that $\theta_0 < c_G - \epsilon_G < c_G + \epsilon_G < b$ and $A_{\epsilon_G} > 0$. Then for $0 < \epsilon < \epsilon_G$, $A_\epsilon \geq A_{\epsilon_G} > 0$.

Lemma 3.1. *Let $\bar{a}$ and $\bar{w}$ be the supermum values of $a(x)$ and $-w'(x)$ on $[c_G - \epsilon_G, c_G + \epsilon_G]$ respectively. Then*

$$R(G, \delta_n) - R(G, \delta_G) \leq (\theta_0 + \mu_G)\epsilon_G^{-4} E(c_n - c_G)^4 + 1/2\bar{a}\bar{w}E(c_n - c_G)^2. \tag{3.6}$$

Lemma 3.2. *Let $M = B^2 c_G^2\{3 + 16B_r[a(c_G)]^2\}$. Then*

(3.7.1) $\lim_{n\to\infty} n^{\frac{2r}{2r+1}} E(c_n - c_G)^2 \le M/[a(c_G)w'(c_G)]^2$;   (3.7.2) $\lim_{n\to\infty} n^{\frac{2r}{2r+1}} E(c_n - c_G)^4 = 0$.

The proofs of Lemma 3.1 and Lemma 3.2 are given in Section 5. Note that, as $\epsilon_G \to 0$, $\bar{a}\bar{w} \to a(c_G)|w'(c_G)|$. Therefore the previous two lemmas give the following theorem.

Theorem 3.1. *Let M be the number defined in Lemma 3.2. Then*

$$\lim_{n\to\infty} n^{\frac{2r}{2r+1}}[R(G,\delta_n) - R(G,\delta_G)] \le M/[2a(c_G)|w'(c_G)|]. \tag{3.8}$$

To consider the uniform convergence rate of δ_n, we define a class of prior distributions. Denote

$$\mathcal{G} = \{G : G \text{ satisfies } \mu_G \le \mu_0, (2.3), c_0 \le c_G \le \rho_0, \min_{x\in[\bar{c}_0,\bar{\rho}_0]} |w'(x)| \ge L\}, \tag{3.9}$$

where $\mu_0 < \infty$, $\theta_0 < c_0 < \rho_0 < b$, $\bar{c}_0 = (c_0 + \theta_0)/2$, $\bar{\rho}_0 = (2\rho_0) \wedge ((\rho_0 + b)/2)$ and $L > 0$. Assume that $\mathcal{G}$ is not empty in the following.

Theorem 3.2. *For some $l > 0$,*

$$\sup_{G\in\mathcal{G}}[R(G,\delta_n) - R(G,\delta_G)] \le l \cdot n^{-\frac{2r}{2r+1}} \tag{3.10}$$

4. A lower bound. We shall obtain a minimax lower bound for the regrets of all monotone empirical Bayes tests first. In the following parts of this paper, l_1, l_2, $\cdots$ stand for the positive constants, which may have different values on different occasions.

Let $\mathcal{C}$ be the set of all estimators c_n^* with $c_n^* \ge 0$ and let $\mathcal{D}$ be the set of all empirical Bayes rules of type (2.1) with $c_n = c_n^* \in \mathcal{C}$. Let $\mathcal{F} = \{f_G(x) = \int f(x|\theta)dG(\theta) : G \in \mathcal{G}\}$ and

c_f be the critical points corresponding to $f \in \mathcal{F}$.

Lemma 4.1.

$$\inf_{\delta_n^* \in \mathcal{D}} \sup_{G \in \mathcal{G}} [R(G, \delta_n^*) - R(G, \delta_G)]$$
$$\geq l_1 \sup \{(c_{f_1} - c_{f_2})^2 : \int [\sqrt{f_1(x)} - \sqrt{f_2(x)}]^2 dx \leq l_2/n, \ \ f_1, f_2 \in \mathcal{F}\}.$$

Suppose that $G_1 \in \mathcal{G}$ with density $g_1(\theta)$ and $c_{f_1} \in (c_0, \rho_0)$. Let $g_2(\theta) = (1+u_n^r \mu_n)^{-1}[g_1(\theta) + u_n^{r-1} A(\theta) H(\frac{\theta - c_{f_1}}{u_n})] I_{[\theta>0]}$, where $\mu_n = \int_{-1}^{1} A(c_{f_1} + t u_n) H(t) dt$ and $H(t)$ is a function such that (1) it has support $[-1, 1]$, (2) $\int_{-1}^{1} H(t) dt = 0$ and $\int_{-1}^{0} H(t) dt \neq 0$, and (3) it has bounded derivatives upto order r. Let $f_i(x) = a(x) \int_0^x \frac{g_i(\theta)}{A(\theta)} d\theta$ for $i = 1, 2$.

Lemma 4.2. *As n is large, $f_2 \in \mathcal{F}$,*

$$\int [\sqrt{f_1(x)} - \sqrt{f_2(x)}]^2 dx \leq l_2/n \quad \textit{and} \quad (c_{f_1} - c_{f_2})^2 \geq l_3 n^{-\frac{2r}{2r+1}}.$$

Theorem 4.1. *For some $l > 0$,*

$$\inf_{\delta_n^* \in \mathcal{D}} \sup_{G \in \mathcal{G}} [R(G, \delta_n^*) - R(G, \delta_G)] \geq l \cdot n^{-\frac{2r}{2r+1}}.$$

Theorem 4.1 says that $n^{-\frac{2r}{2r+1}}$ is the best possible rate of convergence. With the result in (3.10), we conclude that $n^{-\frac{2r}{2r+1}}$ is the optimal rate of monotone empirical Bayes tests and δ_n defined by (3.4) achieves this rate. So δ_n has good performance not only for small samples but also for large samples.

5. Proofs.

5.1. Proof of Lemma 3.1. From (2.2),

$$\begin{aligned} R(G,\delta_n)-R(G,\delta_G) &\le E[I_{[|c_n-c_G|>\epsilon_G]}\int_{c_n}^{c_G} w(x)a(x)dx] + \bar{a}E[I_{[|c_n-c_G|\le\epsilon_G]}\int_{c_n}^{c_G} w(x)a(x)dx] \\ &\le (\theta_0+\mu_G)\epsilon_G^{-4}E(c_n-c_G)^4 + 1/2\bar{a}\bar{w}E(c_n-c_G)^2, \end{aligned}$$

where $\int_{c_n}^{c_G} w(x)a(x)dx \le (\theta_0+\mu_G)$ and by Taylor expansion

$$I_{[|c_n-c_G|\le\epsilon_G]}\int_{c_n}^{c_G} w(x)dx = -1/2\times w'(\hat{c}_n)(c_n-c_G)^2 I_{[|c_n-c_G|\le\epsilon_G]} \le 1/2\bar{w}(c_n-c_G)^2.$$

5.2. Proof of Lemma 3.2. Recall $A_\epsilon>0$ for $\epsilon<\epsilon_G$. For $\epsilon<\epsilon_G$, Let $\eta_1=c_G-\epsilon$ and $\eta_2=c_G+\epsilon$. Rewrite $W_n(x)=\frac{1}{n}\sum_{j=1}^n V_n(X_j,x)$, where

$$V_n(X_j,x)=\frac{\theta_0-x}{u_n}\times\frac{K_0(\frac{x-X_j}{u_n})}{a(X_j)}-\frac{K_1(\frac{x-X_j}{u_n})}{a(X_j)}.$$

Note that $V_n(X_j,x)$ are i.i.d. with for fixed x and n. Let $w_n(x)=E[V_n(X_j,x)]$, $Z_{jn}=V_n(X_j,x)-w_n(x)$, $\sigma_n^2=EZ_{jn}^2$ and $\gamma_n=E[|Z_{jn}|^3]$. Denote $p_n=\min\{a(x): x\in[\eta_1-u_n,\eta_2]\}$. Then we have the following lemma. Its proof is given in the subsection 5.5.

Lemma 5.1. *The following statements hold for large n.*

(i) *For* $x\in[\theta_0,\eta_1)\cup(\eta_2,b)$, $|w(x)|\ge\epsilon A_\epsilon$ *as* $\epsilon<\epsilon_G$;

For $x\in[\theta_0,d_n]$, $|w(x)|\le(2\theta_0+\ln n)B_r$.

(ii) *For all* $x\in[\theta_0,d_n]$, $|w_n(x)-w(x)|\le B_rBu_n^rx\equiv 1/2\beta(x)$.

(iii) *For* $x\in[\eta_1,\eta_2]$, $\sigma_n^2\le m(p_nu_n)^{-1}$, $m=(\eta_2-\theta_0+u_n)^2B^2B_r$;

For $x\in[\theta_0,d_n]$, $\sigma_n^2\le l_2(\ln n)^2u_n{}^{-4/3}$.

(iv) *For* $x\in[\eta_1,\eta_2]$, $\gamma_n\le l_3(p_nu_n)^{-2}$; *For* $x\in[\theta_0,d_n]$, $\gamma_n\le l_4(\ln n)^3u_n^{-8/3}$.

(v) *For* $x\in[\theta_0,d_n]$, $w(x)>\beta(x)\Longrightarrow w_n(x)\ge 1/2w(x)$.

(vi) *For* $x\in[\theta_0,d_n]$, $w(x)<-\beta(x)\Longrightarrow w_n(x)\le 1/2w(x)$.

Since $c_G<b$, assume that $c_G<d_n$ for all n without loss of generality. Based on (3.5), we

decompose $c_n - c_G$ as follows:

$$c_n - c_G = -I_1 - I_3 - I_5 + I_2 + I_4 + I_6, \tag{5.1}$$

where

$$I_1 = \textstyle\int_{-\theta_0}^{\eta_1} I_{[W_n(x)\le 0]}dx, \qquad I_2 = \int_{\eta_2}^{d_n} I_{[W_n(x)>0]}dx,$$

$$I_3 = \textstyle\int_{\eta_1}^{c_G} I_{[W_n(x)\le 0, w(x)\le\beta(\eta_1)]}dx, \quad I_4 = \int_{c_G}^{\eta_2} I_{[W_n(x)>0, w(x)\ge -\beta(\eta_2)]}dx,$$

$$I_5 = \textstyle\int_{\eta_1}^{c_G} I_{[W_n(x)\le 0, w(x)>\beta(\eta_1)]}dx, \quad I_6 = \int_{c_G}^{\eta_2} I_{[W_n(x)>0, w(x)<-\beta(\eta_2)]}dx.$$

Note that $E(c_n - c_G)^2 \le 2\{E[d_nI_1 + I_3^2 + I_5^2 + d_nI_2 + I_4^2 + I_6^2]\}$. To prove (3.7.1), we want to show that

$$(5.2.1) \qquad \lim_{\epsilon\to 0}\lim_{n\to\infty} E[d_nI_1 + I_3^2 + I_5^2] \le \frac{M}{4[a(c_G)w'(c_G)]^2},$$

and

$$(5.2.2) \qquad \lim_{\epsilon\to 0}\lim_{n\to\infty} E[d_nI_2 + I_4^2 + I_6^2] \le \frac{M}{4[a(c_G)w'(c_G)]^2}.$$

We only prove (5.2.2). The proof for (5.2.1) is similar. For $w(x) < -\beta(x)$, $w_n(x) < 1/2w(x) < 0$ from (vi) of Lemma 5.1. Then we have

$$P(W_n(x) > 0) = P\Big(\frac{1}{\sqrt{n\sigma_n^2}}\sum_{j=1}^n Z_{jn} > \frac{-\sqrt{n}w_n(x)}{\sigma_n}\Big) \le P\Big(\frac{1}{\sqrt{n\sigma_n^2}}\sum_{j=1}^n Z_{jn} > \frac{-\sqrt{n}w(x)}{2\sigma_n}\Big).$$

Applying Theorem 5.16 on page 168 in Petrov (1995) to the left-hand-side of the above inequality, $P(W_n(x) > 0) \le [1 - \Phi(\frac{\sqrt{n}|w(x)|}{2\sigma_n})] + \frac{8A\gamma_n}{\sqrt{n}[2\sigma_n + \sqrt{n}|w(x)|]^3} \equiv S_n(x) + T_n(x)$, where A is a constant and $\Phi(\cdot)$ is the cdf of $N(0,1)$. For $x \in [\eta_2, d_n]$, $w(x) \le -\epsilon A_\epsilon$. Then $w(x) < -\beta(x)$ as n is large since $u_n^r d_n \to 0$. Then $P(W_n(x) > 0) \le S_n(x) + T_n(x)$. Note that $\sigma_n^2 \le l_2(\ln n)^2 u_n^{-4/3}$ and $\gamma_n \le l_4(\ln n)^3 u_n^{-8/3}$. Then $S_n(x) \le 1 - \Phi(n^{1/4})$ and $T_n(x) \le n^{-1}$ as n is large. Thus

$$d_nE[I_2] = d_n\int_{\eta_2}^{d_n} P(W_n(x) > 0)dx \le d_n^2[1 - \Phi(n^{1/4}) + n^{-1}] = o(n^{-2r/(2r+1)}). \tag{5.3}$$

For $x \in [c_G, \eta_2]$, $|w'(x)| \ge A_\epsilon$. Then $I_4^2 \le A_\epsilon^{-2}[\int_{c_G}^{\eta_2} I_{[w(x)\ge -\beta(\eta_2)]}w'(x)dx]^2 \le A_\epsilon^{-2}[\beta(\eta_2)]^2$ by

letting $y = w(x)/\beta(\eta_2)$. Therefore

$$\lim_{\epsilon \to 0} \lim_{n \to \infty} n^{\frac{2r}{2r+1}} I_4^2 \le [2B_r B c_G / w'(c_G)]^2. \tag{5.4}$$

By Holder inequality,

$$E[I_6^2] \le \int_{c_G}^{\eta_2} P(W_n(x) > 0) |w(x)|^3 I_{[w(x) < -\beta(\eta_2)]} dx \times \int_{c_G}^{\eta_2} |w(x)|^{-3} I_{[w(x) < -\beta(\eta_2)]} dx.$$

Similar to I_4^2, $\int_{c_G}^{\eta_2} |w(x)|^{-3} I_{[w(x) > -\beta(\eta_2)]} dx \le 1/\{2A_\epsilon [\beta(\eta_2)]^2\}$. Since $w(x) \le -\beta(\eta_2) \le -\beta(x)$ for all $x \in [c_G, \eta_2]$, $P(W_n(x) > 0) \le S_n(x) + T_n(x)$ and

$$E[I_6^2] \le 1/\{2A_\epsilon [\beta(\eta_2)]^2\} \cdot [\int_{c_G}^{\eta_2} S_n(x) |w(x)|^3 dx + \int_{c_G}^{\eta_2} T_n(x) |w(x)|^3 dx]. \tag{5.5}$$

For $x \in [c_G, \eta_2]$, $\sigma_n^2 \le m(p_n u_n)^{-1}$, $\gamma_n \le l_3 (p_n u_n)^{-2}$. Therefore

$$\int_{c_G}^{\eta_2} S_n(x) |w(x)|^3 dx \le \frac{1}{A_\epsilon} \int_{c_G}^{\eta_2} [1 - \Phi(\frac{\sqrt{n u_n p_n} |w(x)|}{2\sqrt{m}})] |w(x)|^3 w'(x) dx \le \frac{6m^2}{A_\epsilon (n u_n p_n)^2}, \tag{5.6}$$

and

$$\int_{c_G}^{\eta_2} T_n(x) |w(x)|^3 dx \le 8 A l_3 \epsilon / (n^2 u_n^2 p_n^2). \tag{5.7}$$

Combining (5.5), (5.6) and (5.7), we have

$$\lim_{\epsilon \to 0} \lim_{n \to \infty} n^{\frac{2r}{2r+1}} E[I_6^2] \le 3B^2 c_G^2 / [2a(c_G) w'(c_G)]^2, \tag{5.8}$$

Then (5.2.2) follows (5.3), (5.4) and (5.8). Thus (3.7.1) is proved. (3.7.2) can be proved similarly. The details are omitted here. Now we complete the proof of Lemma 3.2.

5.3. Proof of Lemma 4.1. Denote $\bar{\mathcal{C}} = \{\bar{c}_n = c_n^* \vee c_0 \wedge \rho_0 : c_n^* \in \mathcal{C}\}$. For $c_n^* \in \mathcal{C}$, $\bar{c}_n = c_n^* \vee c_0 \wedge \rho_0 \in \bar{\mathcal{C}}$. Define $\underline{a} = \{a(x) : x \in [c_0, \rho_0]\}$. Then $\underline{a} > 0$ and

$$\int_{c_n^*}^{c_G} w(x) a(x) dx \ge \int_{\bar{c}_n}^{c_G} w(x) a(x) dx \ge \underline{a} \int_{\bar{c}_n}^{c_G} w(x) dx = -\frac{\underline{a}}{2} w'(\hat{\bar{c}}_n)(\bar{c}_n - c_G)^2,$$

where $\hat{\bar{c}}_n$ is an intermediate value between $\bar{c}_n$ and c_G. Clearly, $\hat{\bar{c}}_n \in [c_0, \rho_0]$. Therefore $|w'(\hat{\bar{c}}_n)| \ge L$. Then $\int_{c_n^*}^{c_G} w(x) a(x) dx \ge l_1 (\bar{c}_n - c_G)^2$ and

$$\inf_{c_n^* \in C} \sup_{G \in \mathcal{G}} E[\int_{c_n^*}^{c_G} w(x) a(x) dx] \ge l_1 \inf_{\bar{c}_n \in \bar{C}} \sup_{G \in \mathcal{G}} E(\bar{c}_n - c_G)^2.$$

Note that $\bar{\mathcal{C}} \subset \mathcal{C}$. Using (2.2),

$$\inf_{\delta_n^* \in \mathcal{D}} \sup_{G \in \mathcal{G}} [R(G, \delta_n^*) - R(G, \delta_G)] \geq l_1 \inf_{\bar{c}_n \in \bar{\mathcal{C}}} \sup_{G \in \mathcal{G}} E(\bar{c}_n - c_G)^2 \geq l_1 \inf_{c_n^* \in \mathcal{C}} \sup_{G \in \mathcal{G}} E(c_n^* - c_G)^2.$$

From the results in Donoho and Liu (1991) (Theorem 3.1 and the remark after Lemma 3.3),

$$\inf_{c_n^* \in \mathcal{C}} \sup_{G \in \mathcal{G}} E(c_n^* - c_G)^2 \geq l_1 \sup \{(c_{f_1} - c_{f_2})^2 : \int [\sqrt{f_1(x)} - \sqrt{f_2(x)}]^2 dx \leq l_2/n, f_1, f_2 \in \mathcal{F}\}.$$

Then Lemma 4.1 is proved.

5.4. Proof of Lemma 4.2. Clearly, as n is large, $c_0 < c_{f_1} - u_n < c_{f_1} + u_n < \rho_0$, $g_2(\theta) \geq 0$ and $c_{f_2} \in (c_0, \rho_0)$. Then $f_2 \in \mathcal{F}$. Note that $\int_0^x H((\theta - c_{f_1})/u_n^{-1}) d\theta = 0$ for $x \leq c_0$ or $x \geq \rho_0$, and $f_1(x) = 0 \Longrightarrow x < \theta_0 \Longrightarrow f_2(x) = 0$. Also for $G \in \mathcal{G}$, $\int_0^{c_0} dG(\theta)/A(\theta) > 0$. Then

$$\begin{aligned}
[\sqrt{f_1(x)} - \sqrt{f_2(x)}]^2 &\leq I_{[f_1(x)>0]}[f_1(x) - f_2(x)]^2/f_1(x) \\
&\leq l_1\{u_n^{2r-2}[\int_0^{c_0} \frac{g_1(\theta)}{A(\theta)} d\theta]^{-1} a(x) [\int_{c_0}^x H(\frac{\theta - c_{f_1}}{u_n}) d\theta]^2 + u_n^{2r} \mu_n^2 f_1(x)\}.
\end{aligned}$$

Note that $\mu_n = \int_{-1}^1 A(c_{f_1} + tu_n) H(t) dt = O(u_n)$ and

$$\int_0^\infty a(x) [\int_{c_0}^x H(\frac{\theta - c_{f_1}}{u_n}) d\theta]^2 dx \leq l_3 u_n^3 \int_{-1}^1 a(c_{f_1} + yu_n) [\int_{-1}^y H(t) dt]^2 dy = O(u_n^3).$$

Then we have

$$\int_0^\infty [\sqrt{f_1(x)} - \sqrt{f_2(x)}]^2 dx \leq l_1[O(u_n^{2r+1}) + O(u_n^{2r+2})] \leq l_2/n.$$

On the other hand, we have $[w_2(c_{f_1})]^2 = [w_2(c_{f_2}) - w_2(c_{f_1})]^2 = [w_2'(\hat{c}_{f_1})]^2 (c_{f_2} - c_{f_1})^2$, where $\hat{c}_{f_1}$ is an intermediate value between c_{f_1} and c_{f_2}. It is easy to see that $[w_2'(\hat{c}_{f_1})]^2 \leq 1/l_1$. Then $(c_{f_2} - c_{f_1})^2 \geq l_1 [w_2(c_{f_1})]^2$. Note that

$$[w_2(c_{f_1})]^2 \geq l_2 u_n^{2r-2} [\int_0^{c_{f_1}} (\theta_0 - \theta) H(\frac{\theta - c_{f_1}}{u_n}) d\theta]^2 \geq l_2 u_n^{2r} [\int_{-1}^0 (\theta_0 - c_{f_1} + tu_n) H(t) dt]^2$$

and $\int_{-1}^0 H(t) dt \neq 0$. Therefore $(c_{f_2} - c_{f_1})^2 \geq l_3 n^{-\frac{2r}{2r+1}}$. The proof of Lemma 4.2 is complete now.

5.5. Proof of Lemma 5.1. Noting that $w(x)$ is decreasing on (θ_0, b) and $w(c_G) = 0$, $|w(x)| \geq |w(c_G - \epsilon)| \wedge |w(c_G + \epsilon)|$ for $x \in (\theta_0, \eta_1) \cup (\eta_2, b)$. Since $w'(x) > A_\epsilon$ for $x \in (\eta_1, \eta_2)$, $|w(c_G - \epsilon)| \geq A_\epsilon \epsilon$ and $|w(c_G + \epsilon)| \geq A_\epsilon \epsilon$. Then $|w(x)| \geq A_\epsilon \epsilon$. On the other hand, since $\phi_G(x) \leq x$ and $\alpha_G(x) \leq B_r$, $|w(x)| \leq (2\theta_0 + \ln n)B_r$ for $x \in [\theta_0, d_n]$. Then (i) is obtained.

With loss of generality, assume that $u_n \leq \theta_0/2$ for all n. It is easy to verify that $w(x) = (\theta_0 - x)\alpha_G(x) + \int_0^x \alpha_G(s)ds$. A straight forward computation shows that for $x \in [\theta_0, d_n]$, $|E[V_n(X_j, x)] - w(x)| \leq u_n^r(x - \theta_0 + u_n)B_r B$. Then (ii) is proved. Note that

$$\sigma_n^2 \leq E[V(X_j, x, n)]^2 = \int_0^1 \frac{1}{u_n a(x - u_n t)}[(\theta_0 - x)K_0(t) - u_n K_1(t)]^2 \alpha_G(x - u_n t)dt.$$

Therefore $\sigma_n^2 \leq m(p_n u_n)^{-1}$ for $x \in [\eta_1, \eta_2]$ and $\sigma_n^2 \leq l_2(\ln n)^2 u_n^{-4/3}$ for $x \in [\theta_0, d_n]$. The results for γ_n can be proved similarly. This completes the proofs of (iii) and (iv).

For $x \in [\theta_0, d_n]$, if $w(x) > \beta(x)$,

$$\frac{w_n(x)}{w(x)} = \frac{w(x) + [w_n(x) - w(x)]}{w(x)} \geq \frac{w(x) - \beta(x) + 1/2\beta(x)}{w(x) - \beta(x) + \beta(x)} \geq \frac{1}{2}.$$

Then (v) is proved. (vi) can be proved in a similar way.

References.

[1] Balakrishnan, N. and Ma, Y. (1997). Convergence rates of empirical Bayes estimation and selection for exponential populations with location parameters. *Advances in Statistical Decision Theory and Applications* (Ed. S. Panchapakesan and N. Balakrishnan) Birkhauser.

[2] Datta, S. (1994). Empirical Bayes estimation in a threshold model. *Sankhya Ser. A* **56**, 106-117.

[3] Donoho, D. L. and Liu, R. C. (1991). Geometrizing rates of convergence. II. *Ann. Statist.* **19**, 633-667.

[4] Huang, S. Y. (1995). Empirical Bayes testing procedures in some nonexponential families using asymmetric Linex loss function. *J. Statist. Plann. Inference* **46**, 293-309.

[5] Huang, S. Y. and Liang T. (1997). Empirical Bayes estimation of the truncation parameter with linex loss. *Statistica Sinica.* **7**, 755-769.

[6] Liang, T. (1993). Convergence rates for empirical Bayes estimation of the scale parameter in a Pareto distribution. *Comput. Statist. Data Anal.* **16**, 35-46.

[7] Ma, Y. and Balakrishnan, N. (2000). Empirical Bayes estimation for truncation parameters. *J. Statist. Plann. Inference* **84**, 111-120.

[8] Prasad, B. and Singh, R. S. (1990). Estimation of prior distribution and empirical Bayes estimation in a nonexponential family. *J. Statist. Plann. Inference* **24**, 81-86.

[9] Petrov, V. V. (1995). *Limits Theorems of Probability Theory.* Clarendon Press · Oxford.

[10] Robbins, H. (1964). The empirical Bayes approach to statistical decision problems. *Ann. Math. Statist.* **35**, 1-20.

[11] Van Houwelingen, J. C. (1976). Monotone empirical Bayes tests for the continuous one-parameter exponential family. *Ann. Statist.* **4**, 981-989.